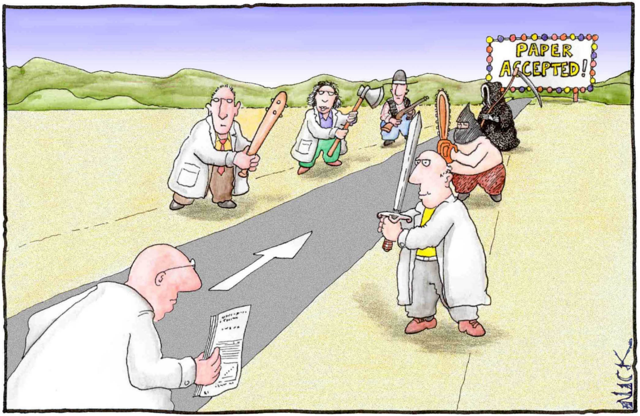
Lower Bounds for Possibly Divergent Probabilistic Programs

Mingshuai Chen

—Joint work with S. Feng, H. Su, B. L. Kaminski, J.-P. Katoen, and N. Zhan—

ROCKS · Nijmegen · May 2022

Work in Progress ...



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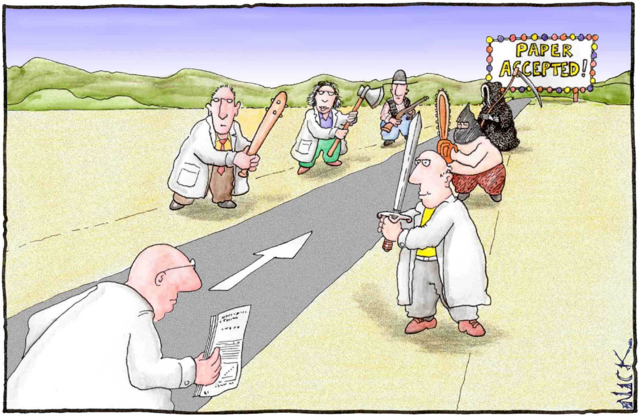

Your Program Diverges? Follow My Rule!

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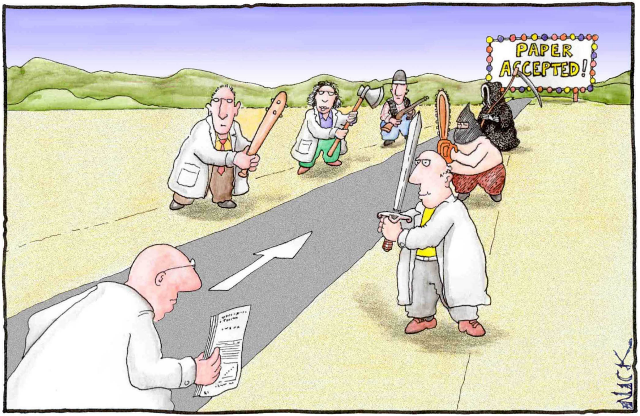

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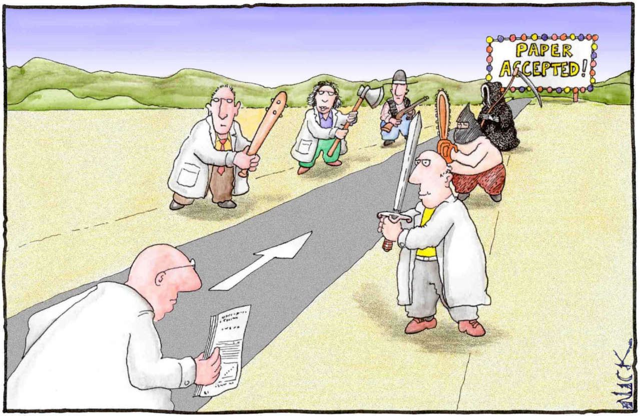

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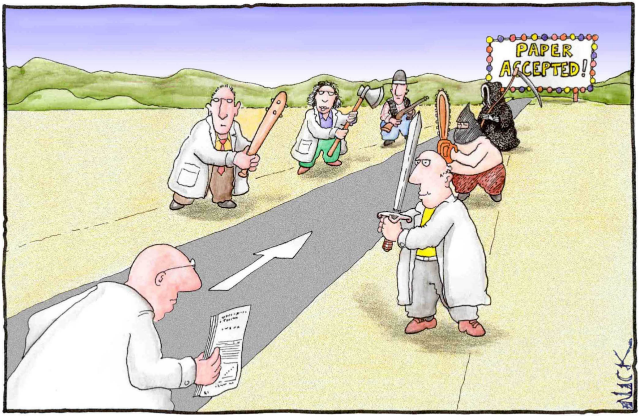

Lower Bounds for Possibly Divergent Probabilistic Programs

Probabilistic Programs

$$C_{\mathsf{brw}}: \quad \texttt{while}\,(\,n > 0\,)\,\{\,n \coloneqq n - 1\;[0.3]\;n \coloneqq n + 1\,\}$$

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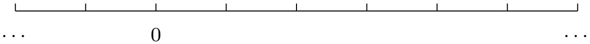

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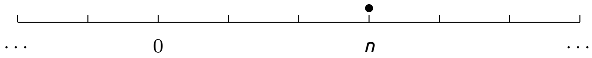

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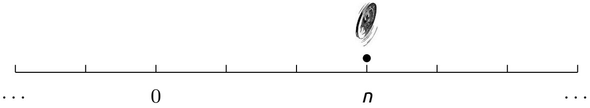

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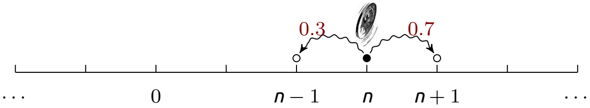

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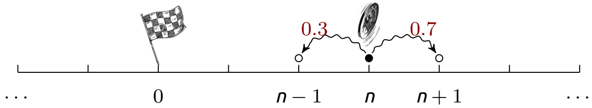

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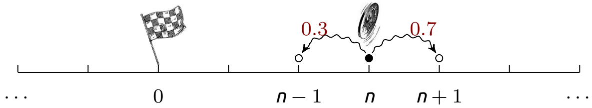
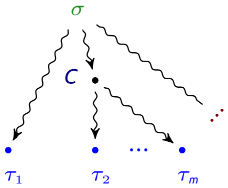
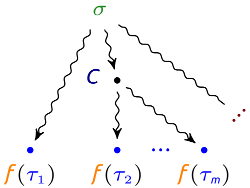


"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

Quantitative Reasoning about Probabilistic Loops [Kozen; McIver, Morgan; Kaminski]




σ
C
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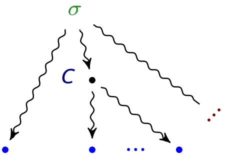
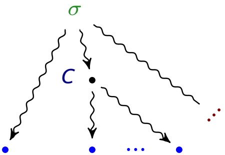
Quantitative Reasoning about Probabilistic Loops [Kozen; McIver, Morgan; Kaminski]



$$\mathsf{wp}[\![C]\!](f)(\sigma) \triangleq \mathsf{Exp}\left[\, f(\tau_1) \quad f(\tau_2) \quad f(\tau_m) \,\right]$$



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$$\mathrm{wp}[\![\mathtt{while}\,(\,\varphi\,)\,\{\, C \,\}]\!]\,(f) \;=\; \mathrm{lfp}\; \Phi_f \;=\; ?$$

Bounding the Least Fixed Point

$$l \;\preceq\; \mathsf{lfp}\,\Phi_f \;\preceq\; u$$

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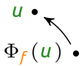

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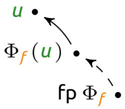

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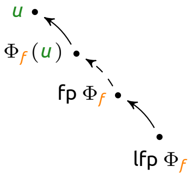

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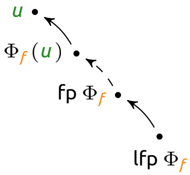

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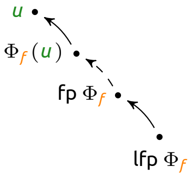

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  $\Phi_f(u) \preceq u$ implies $\mathrm{lfp}\, \Phi_f \preceq u$.

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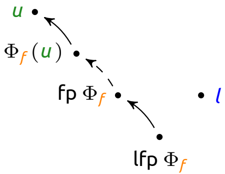

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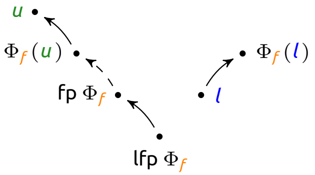

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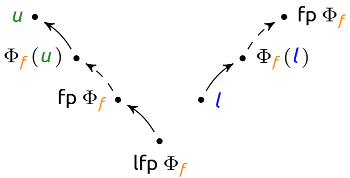

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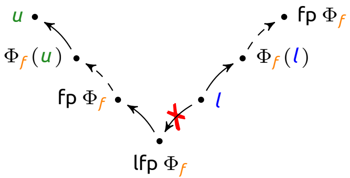

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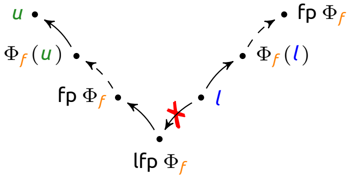
$$l \;\preceq\; \Phi_f(l) \;\wedge\; l \text{ is uni. int.} \quad \text{implies} \quad l \;\preceq\; \mathsf{lfp}\,\Phi_f\,.$$



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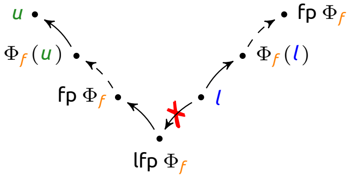
almost-sure termination

bounded expectations

...



A New Proof Rule for Lower Bounds

$$\textit{Loop}\colon\quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \quad\rightsquigarrow\quad \textit{Loop}'\colon\quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

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- Applicable to *possibly divergent Loop*.

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- Applicable to *possibly divergent Loop*.
- l can be *arbitrarily tight*.
- Reducible to *probabilistic BMC*.
- Easier to ensure *uni. int.* for Loop'.
- ...

Example : Biased Random Walk

C_{brw} : $\texttt{while}\,(0 < n)\,\{$
$\quad n := n - 1\ [0.3]\ n := n + 1$
$\}$

C_{brw}^{M} : $\texttt{while}\,(0 < n < M)\,\{$
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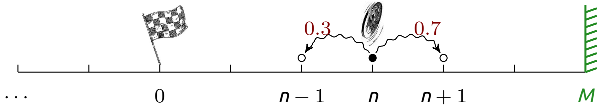

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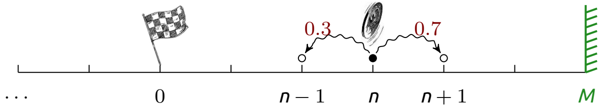

$$l_M(n) \;=\; (3/7)^n - (3/7)^M \,.$$

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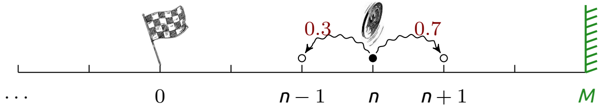

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$$\forall M \in \mathbb{N}\colon \quad l_M \ \preceq\ \mathrm{wp}[\![C^{M}_{\text{brw}}]\!]\,([n \leq 0] \cdot 1) \ \preceq\ \mathrm{wp}[\![C_{\text{brw}}]\!]\,(1)\ .$$

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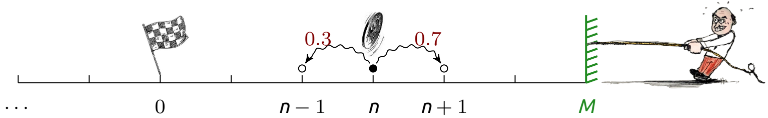

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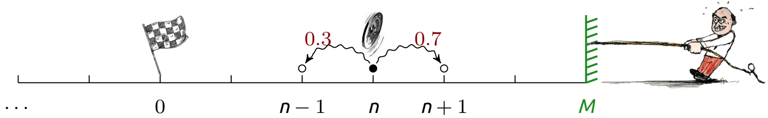

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