Lower Bounds for Possibly Divergent Probabilistic Programs

Shenghua Feng, **Mingshuai Chen**, Han Su, Benjamin L. Kaminski,
Joost-Pieter Katoen, Naijun Zhan

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A Fun Fact

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— Shizuo Kakutani

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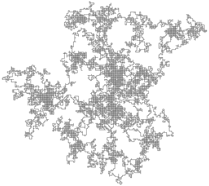

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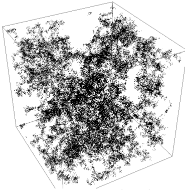

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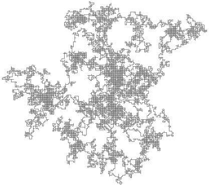

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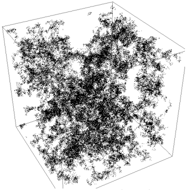

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Question : How to compute *sound approx.* of the returning probability of the bird ?

Probabilistic Programs

$$C_{\mathsf{brw}}: \quad \texttt{while}\,(\,n > 0\,)\,\{\,n \coloneqq n - 1\ [1/3]\ n \coloneqq n + 1\,\}$$

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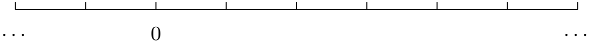

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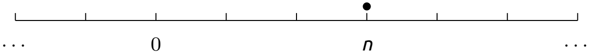

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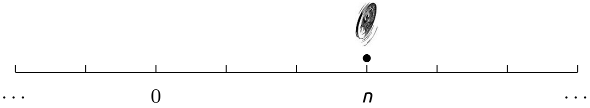

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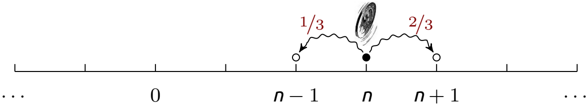

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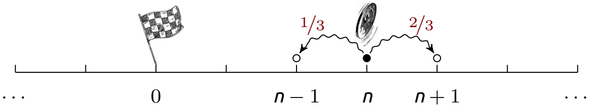

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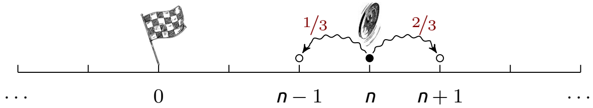
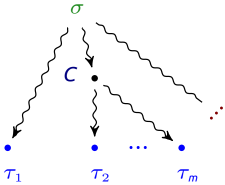


"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

Quantitative Reasoning about Probabilistic Loops [Kozen; McIver, Morgan; Kaminski]



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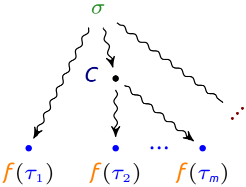

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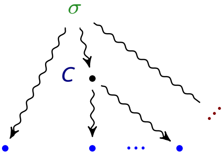

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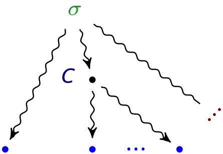

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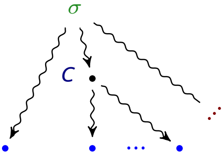

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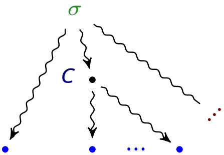

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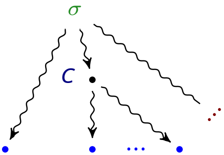

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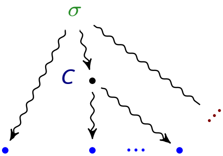

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$$l \preceq \mathrm{lfp}\,\Phi_f \preceq u$$

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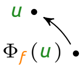

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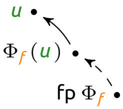

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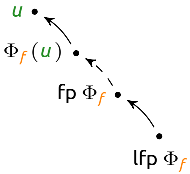

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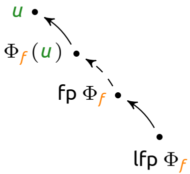

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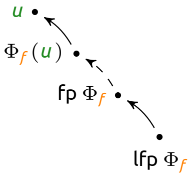

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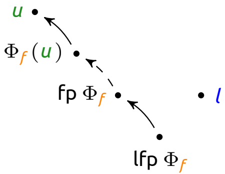

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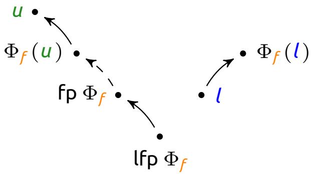

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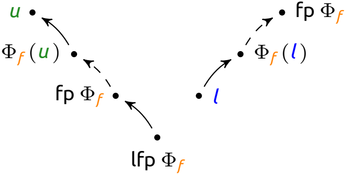

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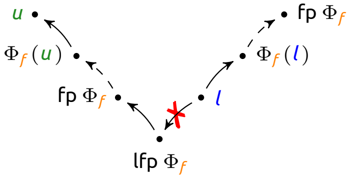

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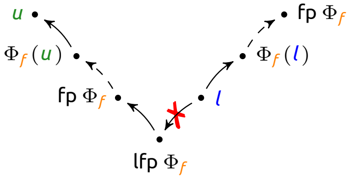

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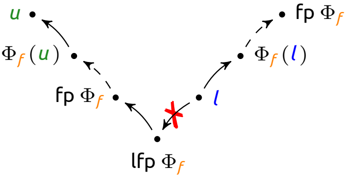
almost-sure termination (AST)
bounded expectations
...



A New Proof Rule for Lower Bounds

Theorem (Guard-Strengthening Rule)

$$C_{\text{loop}}: \quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \qquad \rightsquigarrow \qquad C'_{\text{loop}}: \quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

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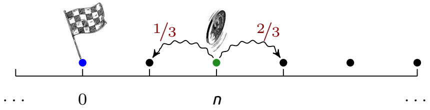

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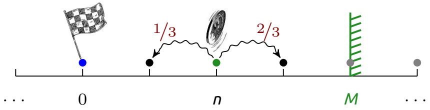

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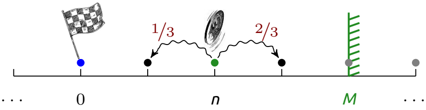

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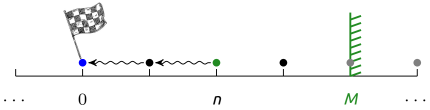

A New Proof Rule for Lower Bounds

Theorem (Guard-Strengthening Rule)

$$C_{\mathsf{loop}}\colon\quad \mathtt{while}\,(\,\varphi\,)\,\{\,C\,\}\qquad\rightsquigarrow\qquad C'_{\mathsf{loop}}\colon\quad \mathtt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathsf{wp}[\![C'_{\mathsf{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathsf{wp}[\![C_{\mathsf{loop}}]\!]\,(f)}\ \text{(Guard-Strengthening)}$$

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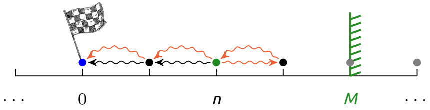

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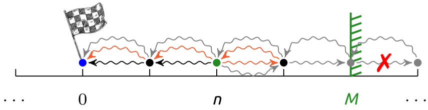

A New Proof Rule for Lower Bounds

Theorem (Guard-Strengthening Rule)

$$C_{\mathsf{loop}} : \quad \texttt{while}\,(\,\varphi\,)\,\{\,C\,\} \quad \rightsquigarrow \quad C'_{\mathsf{loop}} : \quad \texttt{while}\,(\,\varphi'\,)\,\{\,C\,\}$$

$$\frac{\varphi' \Longrightarrow \varphi \qquad l \preceq \mathsf{wp}[\![C'_{\mathsf{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathsf{wp}[\![C_{\mathsf{loop}}]\!]\,(f)}\ \text{(Guard-Strengthening)}$$

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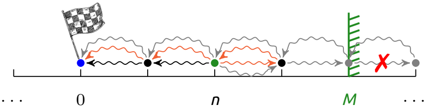

- C'_{loop} features a **stronger** termination property (e.g., becoming AST).
- **Easier** to verify the uni. int. of l and the boundedness of expectations.

Behind the Proof Rule

Theorem (wp-Difference)

$$
\begin{aligned}
\mathsf{wp}[\![C_{\mathsf{loop}}]\!]\,(f) - \mathsf{wp}[\![C'_{\mathsf{loop}}]\!]\,(f) \;=&\\
&\mathsf{wp}[\![\mathtt{while}\,(\,\varphi \wedge \varphi'\,)\,\{\,C\,\}]\!]\,([\neg\varphi \wedge \varphi'] \cdot f) + \lambda\sigma\mathbf{.}\int_A f_{C_{\mathsf{loop}}}\,\mathrm{d}\,({}^{\sigma}\mathbb{P}) -\\
&\mathsf{wp}[\![\mathtt{while}\,(\,\varphi \wedge \varphi'\,)\,\{\,C\,\}]\!]\,([\varphi \wedge \neg\varphi'] \cdot f) - \lambda\sigma\mathbf{.}\int_B f_{C'_{\mathsf{loop}}}\,\mathrm{d}\,({}^{\sigma}\mathbb{P})
\end{aligned}
$$

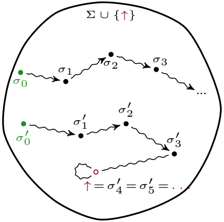

Figure – Infinite prog. traces.

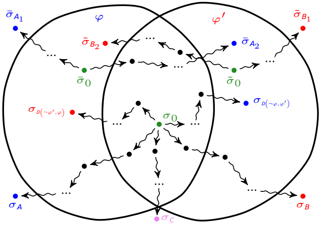

Figure – Illustration of wp-Difference.

- Potentially applicable to *sensitivity analysis* and *model repair*.

Properties of the Proof Rule

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\,([\neg\varphi] \cdot f)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\,(f)} \quad \text{(Guard-Strengthening)}$$

- (Trivially) **complete :** where there's an l, there's a φ' (albeit not "good" enough).

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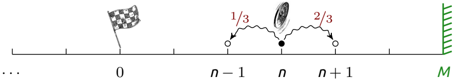
- (Trivially) **complete :** where there's an l, there's a φ' (albeit not "good" enough).
- **General :** applicable to *possibly divergent* C_{loop} and unbounded expectations f, l:



$$l_M = [n<0] + [0\le n\le M]\cdot\Big(\left(1/2\right)^n - \left(1/2\right)^M\Big)$$

$$\forall M\in\mathbb{N}:\quad l_M \overset{\text{Hark}}{\preceq} \mathrm{wp}[\![C^M_{\mathrm{brw}}]\!]\,([n\le 0]\cdot 1) \overset{\text{Stren.}}{\preceq} \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\,(1)$$

Properties of the Proof Rule

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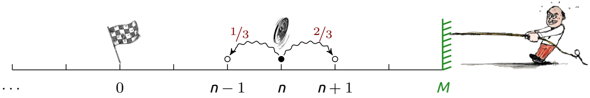

$$l_M \;=\; [n<0] + [0 \le n \le M]\cdot\left((1/2)^n - (1/2)^M\right)$$

$$\forall M \in \mathbb{N}\colon \quad l_M \overset{\text{Hark}}{\preceq} \mathrm{wp}[\![C^M_{\mathrm{brw}}]\!]\,([n\le 0]\cdot 1) \overset{\text{Stren.}}{\preceq} \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\,(1)$$

- **Tight :** the underapproximation error approaches 0 as $\varphi' \to \varphi$:

$$[n<0] + [n \ge 0]\cdot(1/2)^n \;=\; \lim_{M\to\infty} l_M \;\preceq\; \mathrm{wp}[\![C_{\mathrm{brw}}]\!]\,(1)$$

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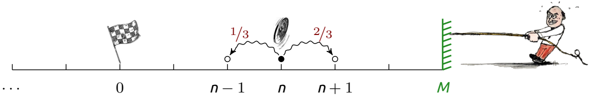

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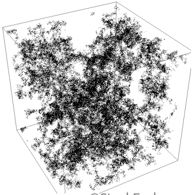
- **Automatable :** reducible to *probabilistic model checking* for finite-state C'_{loop} :

$$\texttt{while}\,(\,x \neq 0 \vee y \neq 0 \vee z \neq 0\,)\,\{$$
$$x \coloneqq x-1 \;\oplus\; x \coloneqq x+1 \;\oplus\; y \coloneqq y-1 \;\oplus\; y \coloneqq y+1 \;\oplus\; z \coloneqq z-1 \;\oplus\; z \coloneqq z+1\,\}$$



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Properties of the Proof Rule

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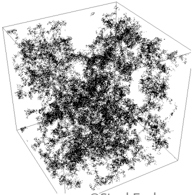
```
while ( $x \neq 0 \vee y \neq 0 \vee z \neq 0$ ) {
    $x \coloneqq x-1 \oplus x \coloneqq x+1 \oplus y \coloneqq y-1 \oplus y \coloneqq y+1 \oplus z \coloneqq z-1 \oplus z \coloneqq z+1$ }
```



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$$\mathcal{P} = 1 - \left(\frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{3-\cos x-\cos y-\cos z}\right)^{-1} = 0.3405373296\ldots$$

Properties of the Proof Rule

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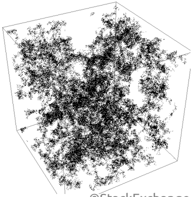

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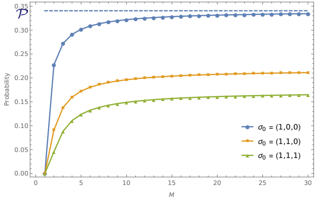

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A “Real” Application : Zeroconf Protocol [Bohnenkamp et al. 2003]

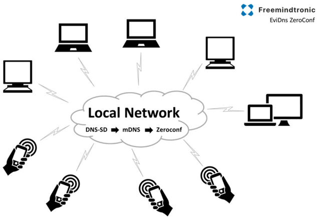

Figure – Self-configuring IP network.

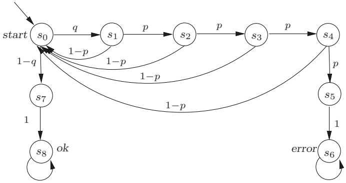

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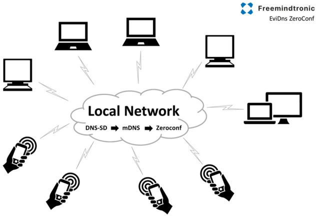
Figure – Markov-chain snippet ($N = 4$).

```
$C_{zc}$ :   $start = 1 \,⨾\, established = 0 \,⨾\, probe = 0 \,⨾$
             while ( $start \leq 1 \wedge established \leq 0 \wedge probe < N \wedge N \geq 4$ ) {
               if ( $start = 1$ ) {
                 { $start := 0$ } [0.5] { $start := 0 \,⨾\, established := 1$ } }
               else { { $probe := probe + 1$ } [0.001] { $start := 1 \,⨾\, probe := 0$ } } }
```

A "Real" Application : Zeroconf Protocol [Bohnenkamp et al. 2003]



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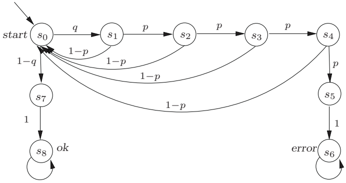

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Pr("starting within the loop guard, C_{zc} terminates with $established = 1$") ≥ 0.99999999999

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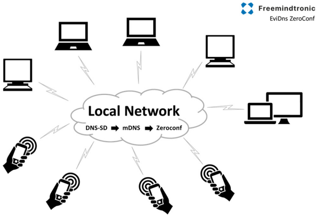

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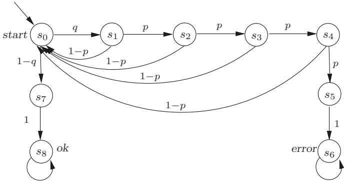

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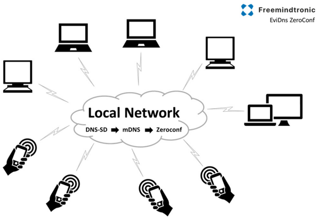
```

Pr(“starting within the loop guard, C_{zc} terminates with *established* $= 1$”) $\overset{?}{\geq} 0.99999999999$

A "Real" Application : Zeroconf Protocol [Bohnenkamp et al. 2003]



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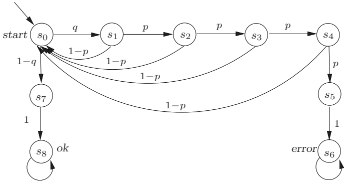

©[Baier & Katoen 2008]

Figure – Markov-chain snippet ($N = 4$).

```
$C_{zc}$ :   $start = 1 \fatsemi established = 0 \fatsemi probe = 0 \fatsemi$
             while ( $start \leq 1 \wedge established \leq 0 \wedge probe < N \wedge N \geq 4 \wedge N \leq 10$ ) {
               if ( $start = 1$ ) {
                 { $start := 0$ } [0.5] { $start := 0 \fatsemi established := 1$ } }
               else { { $probe := probe + 1$ } [0.001] { $start := 1 \fatsemi probe := 0$ } } }
```

Pr("starting within the loop guard, C_{zc} terminates with *established* $= 1$") $\overset{\checkmark}{\geq} 0.99999999999$

Summary

$$\frac{\varphi' \implies \varphi \qquad l \preceq \mathrm{wp}[\![C'_{\mathrm{loop}}]\!]\,([\neg\varphi]\cdot f)}{l \preceq \mathrm{wp}[\![C_{\mathrm{loop}}]\!]\,(f)}\ \text{(Guard-Strengthening)}$$

- a new lower bound rule based on wp-difference and guard-strengthening;

⇒ Feng, Chen, Su, Kaminski, Katoen, Zhan : *Lower Bounds for Poss. Divergent Prob. Prog.* OOPSLA ’23.

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- tight lower bounds for 3-D random walks on $\mathbb{Z}^3$ and the Zeroconf protocol.

More in the paper :

- how to find a "good" strengthening $\varphi' \implies \varphi$?
- how to generate a non-trivial lower bound for C'_{loop}?
- corner cases where guard strengthening is insufficient;
- ...

⇒ Feng, Chen, Su, Kaminski, Katoen, Zhan : *Lower Bounds for Poss. Divergent Prob. Prog.* OOPSLA '23.