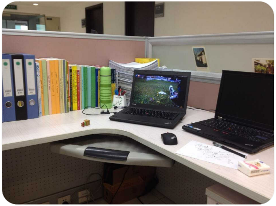
Latticed *k*-Induction

with an Application to Probabilistic Programs

Mingshuai Chen

—Joint work with K. Batz, B. L. Kaminski, J.-P. Katoen, C. Matheja, P. Schröer—

My Ph.D. at ISCAS



first day onboard

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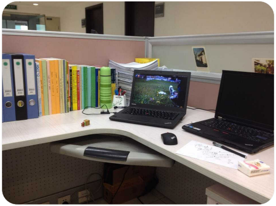

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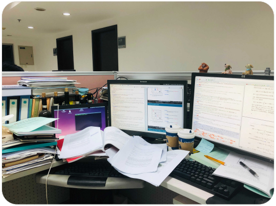

last day finishing the thesis

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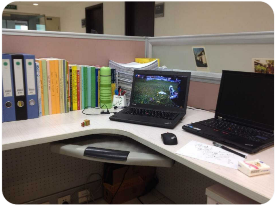

first day onboard

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my Ph.D. life in between

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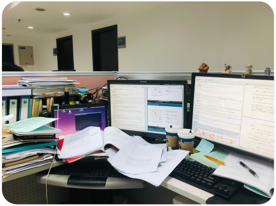

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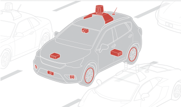
Cyber-Physical Systems (CPS)

An open, interconnected form of embedded systems that integrates capabilities of *computing*, *communication*, and *control*, among which many are safety-critical.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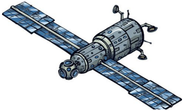
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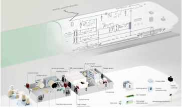
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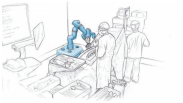
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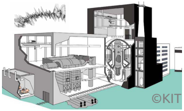
robot surgeon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robust control

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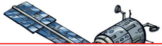


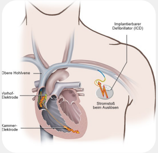

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(USA, 1997 – 2003)

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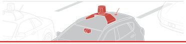
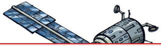


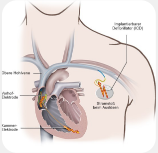

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"How can we provide people with CPS they can bet their lives on?"

— Jeannette M. Wing, former AD for CISE at NSF

Formal Methods

Joseph Sifakis
2007 Turing Awardee

"[...] the challenge of designing embedded systems offers a unique opportunity for reinvigorating computer science. The challenge, and thus the opportunity, spans the spectrum from theoretical foundations to engineering practice. To begin with, we need a mathematical basis for systems modeling and analysis which integrates both computation and physical constraints in a consistent, operative manner [...]"

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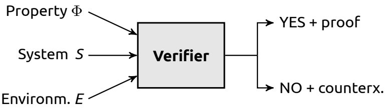

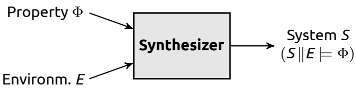

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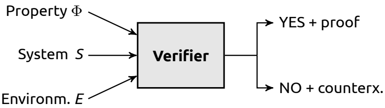

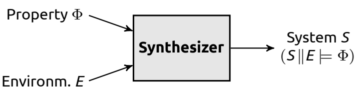

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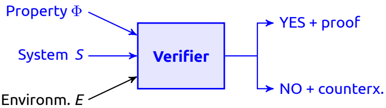

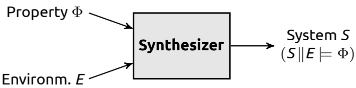

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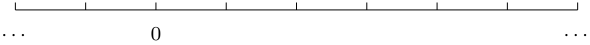

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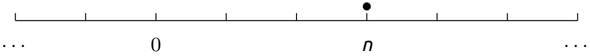

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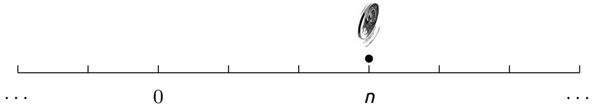

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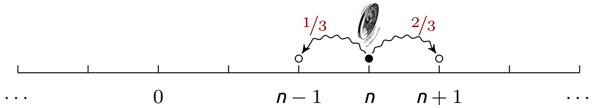

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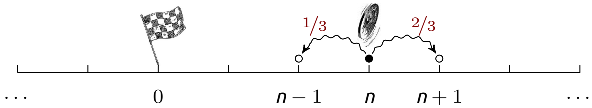

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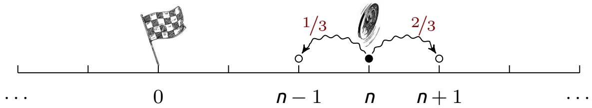

"The crux of probabilistic programming is to treat normal-looking programs as if they were probability distributions."

— Michael Hicks, The PL Enthusiast

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$\Downarrow$ **Latticed k-Induction**

Yes. It enables fully automatic verification of non-trivial properties.

k-Induction for Probabilistic Programs – Intuition

For a probabilistic loop C_{geo} :

$$\texttt{while}\,(\,c = 1\,)\,\{\,c \coloneqq 0\,[\,1/2\,]\,x \coloneqq x + 1\,\}\,,$$

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is not inductive but 2-inductive.

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Given : $\mathsf{TS} = (S, I, T)$, invariant property $P \subseteq S$.

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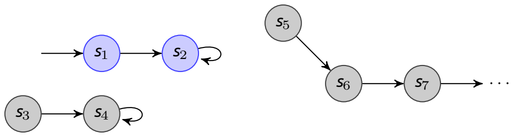

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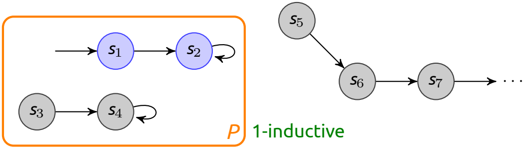

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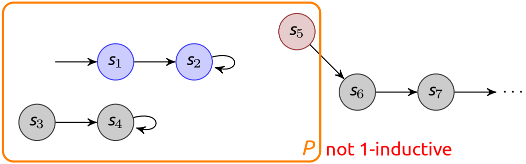

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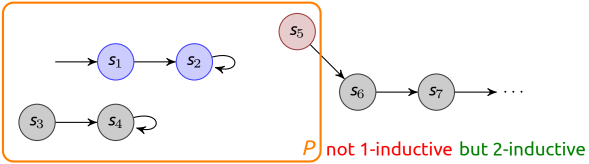

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For verifying *probabilistic programs*, we have to

- leave the Boolean domain and reason about quantities;

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$$\underbrace{I(s_1) \wedge T(s_1, s_2) \wedge \ldots \wedge T(s_{k-1}, s_k)}_{\text{all states reachable within } k-1 \text{ steps}} \implies \underbrace{P(s_1) \wedge \ldots \wedge P(s_k)}_{\text{are } P\text{-states}} \qquad \text{[base case]}$$

$$\underbrace{P(s_1) \wedge T(s_1, s_2) \wedge \ldots \wedge P(s_k)}_{\text{assume staying in } P \text{ for } k-1 \text{ steps}} \wedge \underbrace{T(s_k, s_{k+1})}_{\text{after step } k} \implies \underbrace{P(s_{k+1})}_{\text{end up in } P \text{ again}} \qquad \text{[ind. step]}$$

then P is a k-inductive invariant covering all reachable states of TS.

For verifying *probabilistic programs*, we have to

- leave the Boolean domain and reason about quantities;
- reason about sets of paths rather than individual paths.

Idea Sketch

k-induction for TS
in terms of a **SAT problem**

Idea Sketch

k-induction for TS
in terms of **fixed points**

$\uparrow$

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Latticed *k*-induction for proving $\mathrm{lfp}\ \Phi \sqsubseteq f$

abstract

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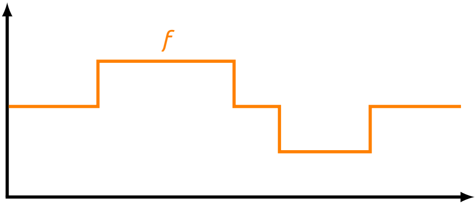

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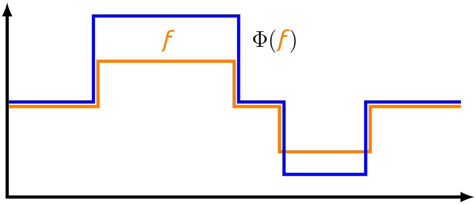

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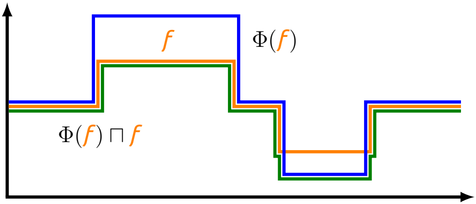

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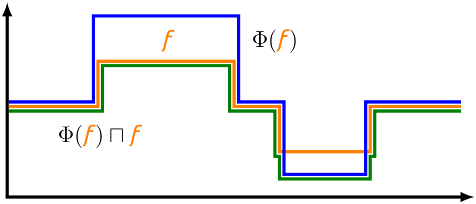

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We call such f k-inductive invariant.

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Can be further generalized to *transfinite κ-induction* (not in this talk).

Key Insights of Soundness

Lemma (Descending chain)

Iterating Ψ_f on f yields a descending chain, i.e.,

$$f \sqsupseteq \Psi_f^1(f) \sqsupseteq \Psi_f^2(f) \sqsupseteq \Psi_f^3(f) \sqsupseteq \ldots .$$

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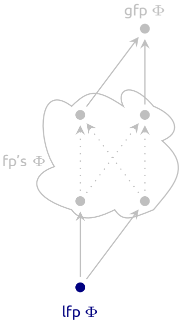

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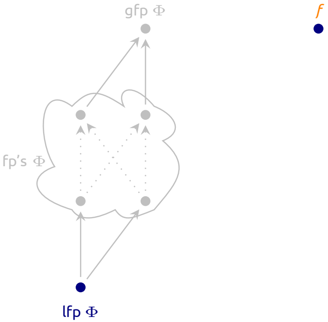

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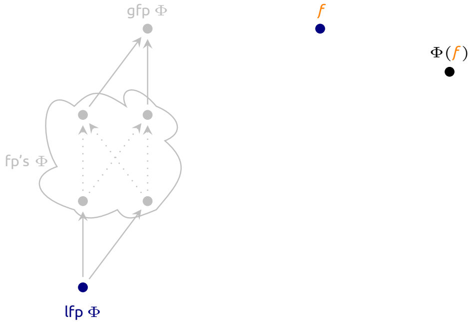

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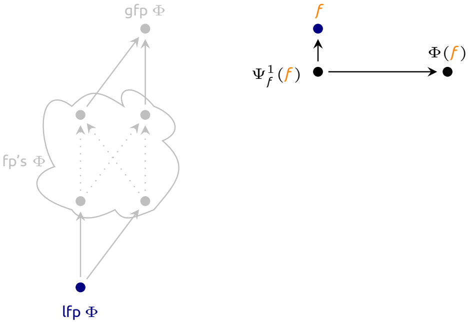

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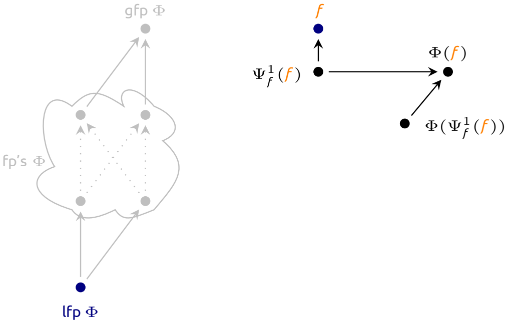

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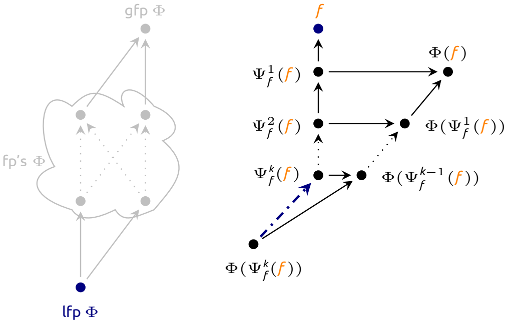

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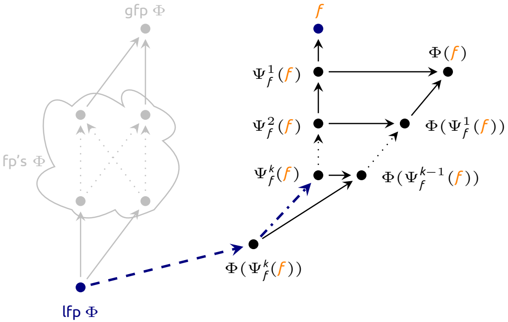

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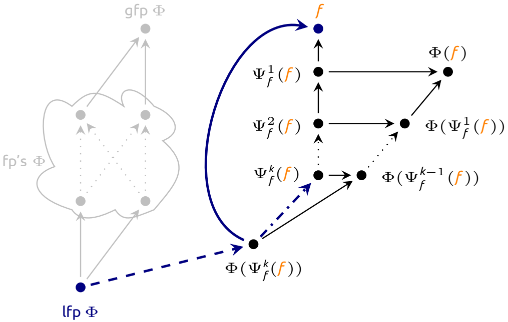

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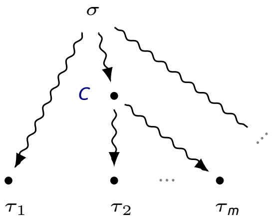

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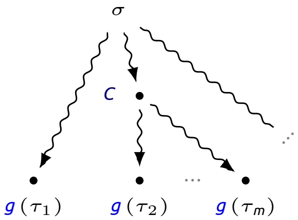

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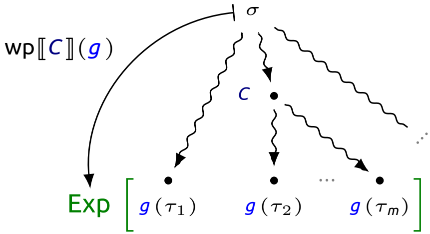

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$$\mathrm{wp}[\![\{\,\mathtt{skip}\,\}\,[\,1/2\,]\,\{\,x := x+2\,\}]\!]\,([x=4]) \;=\; \frac{1}{2}\cdot[x=4] + \frac{1}{2}\cdot[x=2]$$

$$\mathrm{wp}[\![\mathtt{while}\,(\,c=1\,)\,\{\,\{\,c := 0\,\}\,[\,1/2\,]\,\{\,x := x+1\,\}\,\}]\!]\,(x) \;=\; [c=1]\cdot(x+1) + [c\neq 1]\cdot x$$

k-Induction for Probabilistic Programs

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Hence, latticed k-induction applies :

Corollary (k-induction for probabilistic programs)

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Here

$$\Psi_f(h) = \Phi\,(h) \sqcap f \qquad \text{where for } h, h' \in \mathbb{E}, \quad h \sqcap h' = \lambda\sigma\text{.}\; \min\{h(\sigma), h'(\sigma)\}\,.$$

Tool Support

kipro2 : ***k*-Induction for PRObabilistic PROgrams**

`https://github.com/moves-rwth/kipro2`

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For *linear* $C = \texttt{while}\,(\varphi)\,\{C'\}$ and *piecewise linear* f, g, kipro2 *semi-decides* by SMT :

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If $\mathrm{wp}[\![C]\!]\,(g) \not\leq f$, kipro2 finds via *bounded model checking* some $\sigma \in \Sigma$ with

$$\mathrm{wp}[\![C]\!]\,(g)\,(\sigma) > f(\sigma)\,.$$

Example : Geometric Distribution

For C_{geo} given by

$$\mathtt{while}\,(\,c = 1\,)\,\{\,\{\,c := 0\,\}\;[\,1/2\,]\;\{\,x := x + 1\,\}\,\}\,,$$

the property

$$\forall \text{ initial state } \sigma : \quad \mathsf{wp}[\![C_{\mathsf{geo}}]\!]\,(x)\,(\sigma) \;\leq\; \sigma(x) + 1$$

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also hold? No; counterexample by BMC : $\sigma(c) = 1, \sigma(x) = 6$.

Example : Bounded Retransmission Protocol [Helmink et al., TYPES '93]

For C_{brp} given by

$$\begin{aligned}&\texttt{while}\,(\,\mathit{sent} < \mathit{toSend} \wedge \mathit{fail} < \mathit{maxFail}\,)\,\{\\&\quad\{\qquad\qquad\qquad\qquad\}\,[0.9]\,\{\qquad\qquad\qquad\qquad\qquad\qquad\}\\&\}\,,\end{aligned}$$

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also hold? No : $\mathit{toSend} = 6052$, $\mathit{maxFail} = 2$, $\mathit{sent} = 6042$, $\mathit{fail} = 0$, $\mathit{totalFail} = 1$.

Example : Uniform Sampling by Fair-Coin Flips [Lumbroso, arXiv'13]

```
while(running = 0){

    v := 2*v;
    {c := 2*c+1}[0.5]{c := 2*c};
    if(not (v<n)){
        if((not (n=c)) & (not (n<c))){ #terminate
            running := 1
        }{
            v := v-n;
            c := c-n;
        }
    }{
        skip
    }

    #on termination, determine correct index
    if((not (running = 0))){
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For *arbitrary* array of fixed size $n = \{2, 3, 4, 5\}$, we verify

$\Pr(\text{“sample fixed element”}) \leq {}^{1}/_{n}$.

Empirical Results (Partial)

	postexpectation	variant	result	k	#formulae	formulae_t	sat_t	total_t
geo	x	1	ind	2	18	0.01	0.00	0.08
		2	ref	11	103	0.04	0.01	0.09
		3	ref	46	1223	0.39	0.04	0.48
unif_sam	$[c = i]$	1	ind	2	267	0.27	0.02	0.56
		2	ind	3	1402	1.45	0.10	1.81
		3	ind	3	1402	1.48	0.11	1.86
		4	ind	5	40568	47.31	15.70	63.28
		5	TO	–	–	–	–	–

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Summary

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Interesting Related Directions

- Programs with conditioning and/or continuous distributions?
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- Latticed Craig interpolation generalizing McMillan's?
  - ⇒ McMillan : *Interpolation and SAT-Based Model Checking.* CAV '03.
  - ⇒ Chen, Wu, Batz, Schröer, Katoen : *Latticed Craig Interpolation w. an App. to Prob. Prog.* Under dev.
- ...

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COLLEGE OF COMPUTER SCIENCE AND TECHNOLOGY
ZHEJIANG UNIVERSITY

In Combination with Latticed BMC

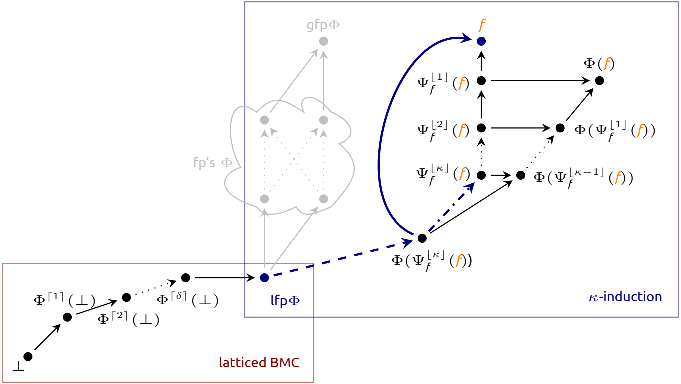

Figure – κ-induction and latticed BMC in case that lfp $\Phi \sqsubseteq f$.