Interpolation over Nonlinear Arithmetic

Towards Program Reasoning and Verification

Mingshuai Chen

—Joint work with J. Wang, B. Zhan, N. Zhan, D. Kapur, J. An, T. Gan, L. Dai, and B. Xia—

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What Is Interpolation?

Interpolation /ɪntəːpəˈleɪʃ(ə)n/

MATHEMATICS

"the insertion of an intermediate value or term into a series by estimating or calculating it from surrounding known values."

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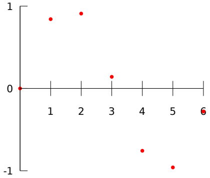

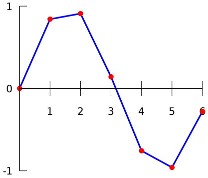

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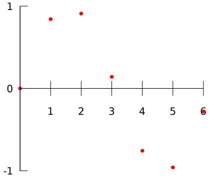

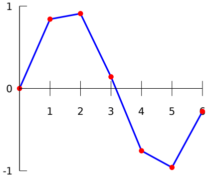

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LOGICAL REASONING

$$P \models Q \qquad\qquad P \models R \models Q$$

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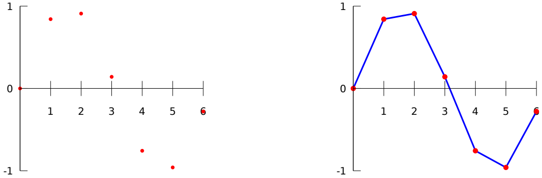

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LOGICAL REASONING

$P \models Q \qquad P \models R \models Q$

$P \wedge Q \models \bot \qquad P \models R$ and $R \wedge Q \models \bot$

Interpolants as Loop Invariants

Example ([Lin et al., ASE '17])

$$\text{while } (x \neq n)\{$$
$$\quad x := x + 1;\ y := y + 1;$$
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assume(x = 0 ∧ y = 0 ∧ n ≥ 0);
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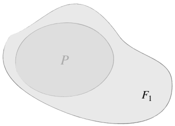

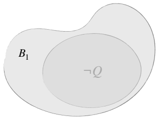

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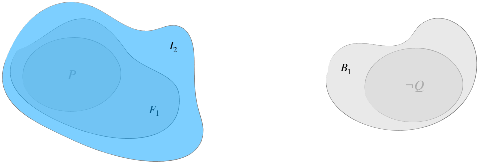

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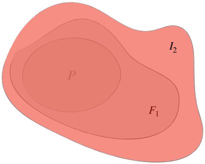

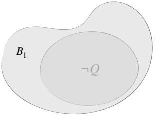

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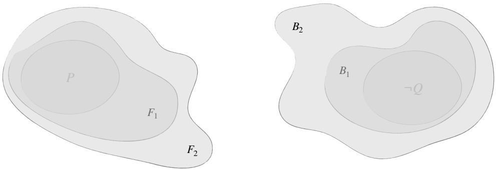

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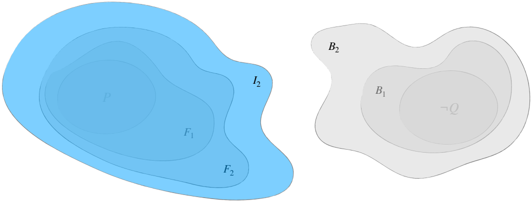

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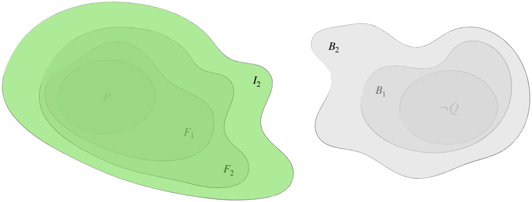

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Given ϕ and ψ in a theory $\mathcal{T}$ s.t. $\phi \wedge \psi \models_{\mathcal{T}} \bot$, I is a *(reverse) interpolant* of ϕ and ψ if

$$\phi \models_{\mathcal{T}} I \quad \text{and} \quad I \wedge \psi \models_{\mathcal{T}} \bot \quad \text{and} \quad \mathit{var}(I) \subseteq \mathit{var}(\phi) \cap \mathit{var}(\psi)\ .$$

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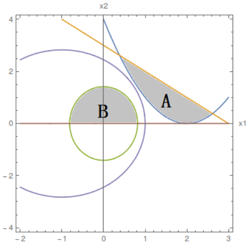
Example (Nonlinear $\mathcal{T}$)

$$A \;\widehat{=}\; -x_1^2 + 4x_1 + x_2 - 4 \geq 0 \wedge -x_1 - x_2 + 3 - y^2 > 0$$

$$B \;\widehat{=}\; -3x_1^2 - x_2^2 + 1 \geq 0 \wedge x_2 - z^2 \geq 0$$

$$I \;\widehat{=}\; -3 + 2x_1 + x_1^2 + \frac{1}{2}x_2^2 > 0$$



Binary Classification

Binary Classifier

Given a dataset $X = X^{+} \uplus X^{-}$ of sample points, $C\colon X \to \{\top, \bot\}$ is a *classifier* if

$$\forall \vec{x} \in X^{+} : C(\vec{x}) = \top \qquad \text{and} \qquad \forall \vec{x} \in X^{-} : C(\vec{x}) = \bot\,.$$

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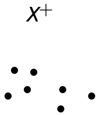

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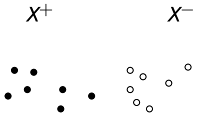

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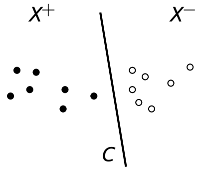

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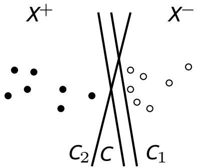

There could be (infinitely) many valid classifiers.

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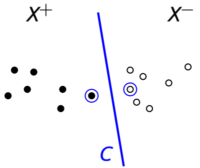

Support Vector Machine (SVM) finds a "middle" one – separating hyperplane that yields the largest distance (functional margin) to the nearest samples (support vectors) – via *convex optimization*.

Interpolation vs. Classification

☺ Linear interpolants can be viewed as hyperplane classifiers [Sharma et al., CAV'12] : sampling from $[\![\phi]\!]$ and $[\![\psi]\!]$ $\rightarrow$ building a hyperplane classifier $\rightarrow$ refining by CEs.

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☹ X^+ and X^- are often not linearly separable for nonlinear ϕ and ψ :

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\begin{aligned}
A \quad \widehat{=} \quad & (x < 2.5 \Rightarrow y \geq 2\sin(x)) \\
& \wedge(x \geq 2.5 \wedge x < 5 \Rightarrow y \geq 0.125x^2 + 0.41) \\
& \wedge(x \geq 5 \wedge x \leq 6 \Rightarrow y \geq 6.04 - 0.5x) \\
B \quad \widehat{=} \quad & (x < 3 \Rightarrow y \leq x\cos(0.1e^x) - 0.083) \\
& \wedge(x \geq 3 \wedge x \leq 6 \Rightarrow y \leq -x^2 + 10x - 22.35)
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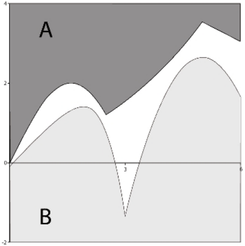

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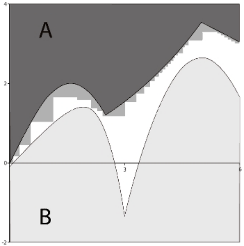

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☺ Encoding interpolants as logical combinations of linear constraints.

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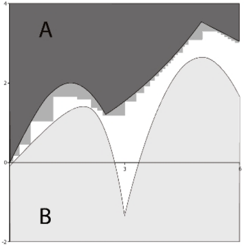
$$A \;\widehat{=}\; (x < 2.5 \Rightarrow y \geq 2\sin(x)) \wedge (x \geq 2.5 \wedge x < 5 \Rightarrow y \geq 0.125x^2 + 0.41) \wedge (x \geq 5 \wedge x \leq 6 \Rightarrow y \geq 6.04 - 0.5x)$$

$$B \;\widehat{=}\; (x < 3 \Rightarrow y \leq x\cos(0.1e^x) - 0.083) \wedge (x \geq 3 \wedge x \leq 6 \Rightarrow y \leq -x^2 + 10x - 22.35)$$



©Kupferschmid & Becker, FORMATS '11

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☹ Yielding rather complex interpolants (even of an infinite length in the worst case).

Interpolation vs. Classification

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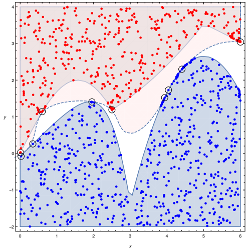

©Chen et al., CADE'19

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☺ NIL : learning nonlinear interpolants.

Space Transformation & Kernel Trick

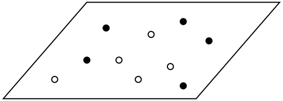
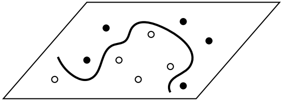


Figure – 2-dimensional input space

Space Transformation & Kernel Trick



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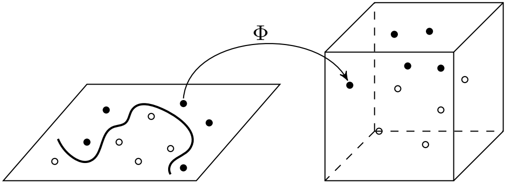

Figure – 2-dimensional input space $\mapsto$ 3-dimensional feature (monomial) space with linear separation.

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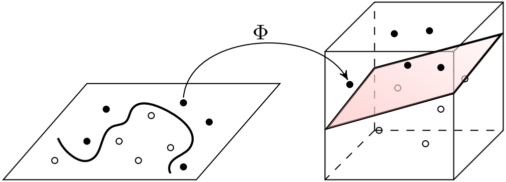

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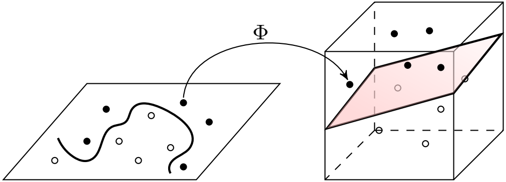

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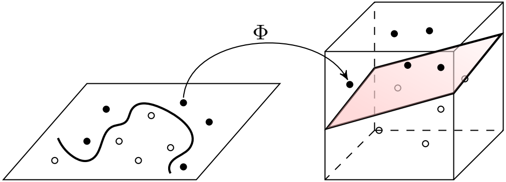

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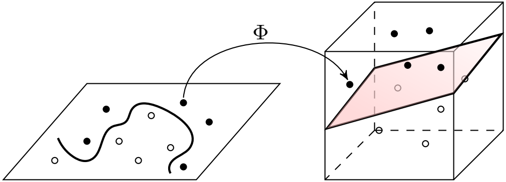

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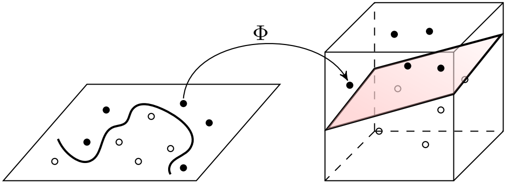

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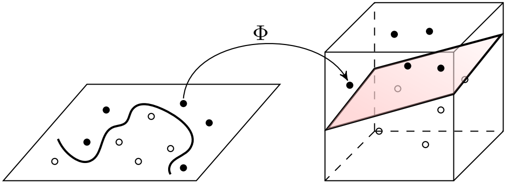

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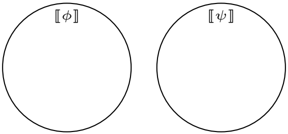
- kernel function (κ)
- support vectors (α_i)
- polynomial degree describing complexity of the monomial space (m)

The NIL Algorithm

1. Given mutually contradictory nonlinear ϕ and ψ over common variables $\mathbf{x}$.
2. Generate sample points by, e.g., (uniformly) scattering random points.
3. Find a classifier by SVMs (with kernel-degree *m*) as a candidate interpolant.
4. Refine the candidate by CEs till it being verified as a true interpolant.



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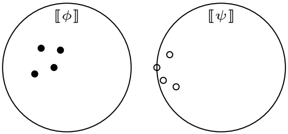

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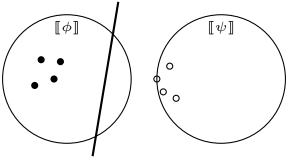

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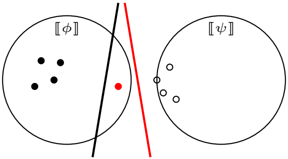

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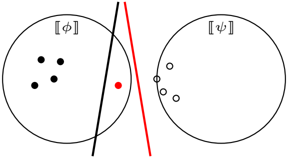

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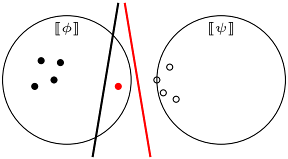

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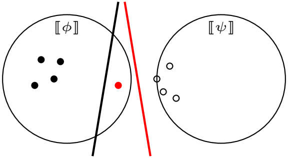

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Comparison with Naïve QE-Based Method

	QE-based method	**NIL**
Logical strength	strongest : $\exists \mathbf{y}.\ \phi(\mathbf{x}, \mathbf{y})$ weakest : $\forall \mathbf{z}.\ \neg\psi(\mathbf{x}, \mathbf{z})$	medium ⇒ robust
Complexity of I	direct projection ⇒ complex	single polynomial ⇒ simple
Efficiency	doubly exponential	$n \times$ doubly exponential

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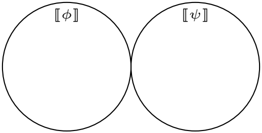
QE + template? ⇒ Too many unknown parameters.

NIL$_\delta$: For Cases with Zero Functional Margin



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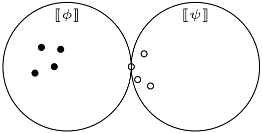

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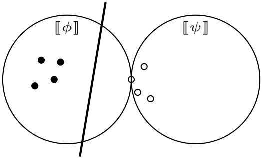
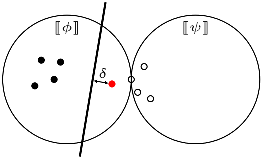


NIL_δ : For Cases with Zero Functional Margin



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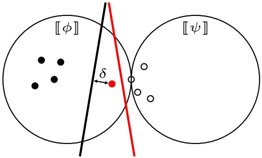

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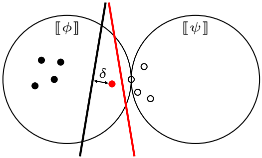

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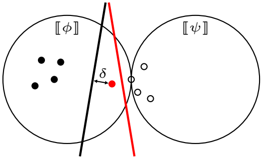

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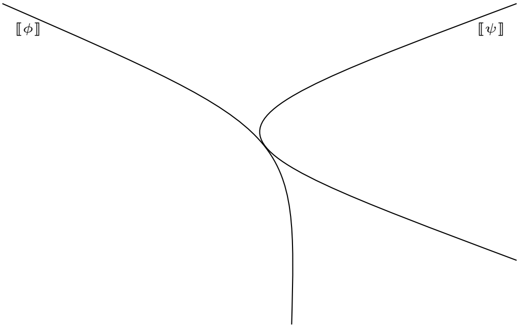
☹ May not converge to an actual interpolant when $[\![\phi]\!]$ or $[\![\psi]\!]$ is unbounded.

NIL$^{*}_{\delta,B}$: For Unbounded Cases with Varying Tolerance



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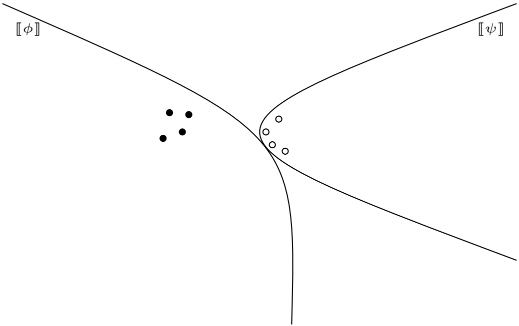

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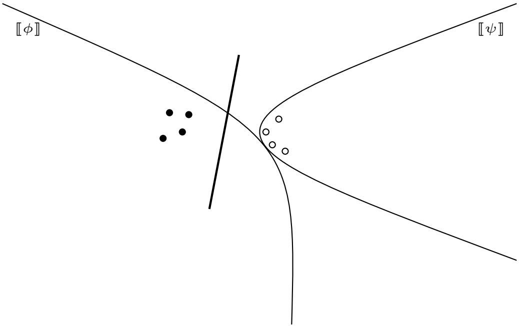

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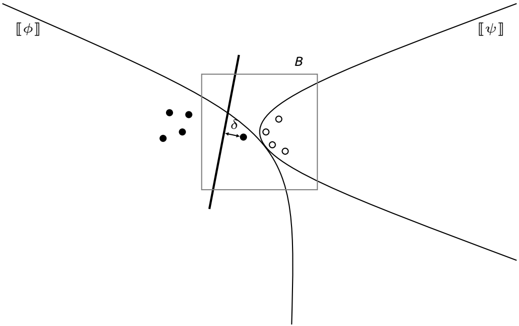

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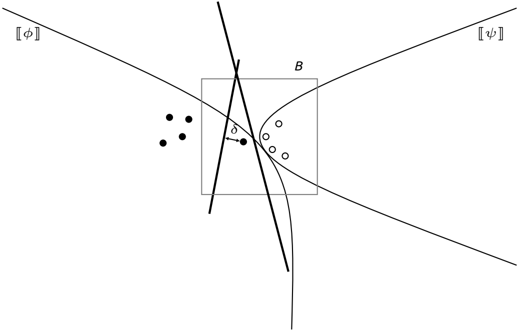

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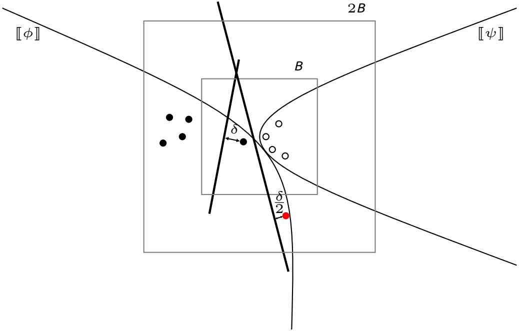

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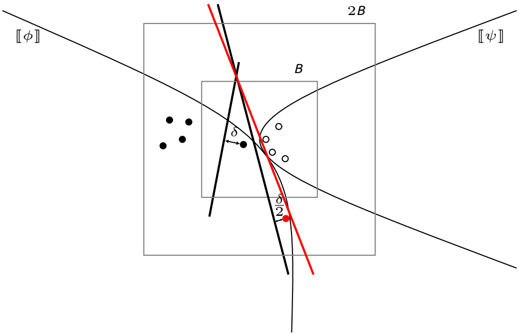

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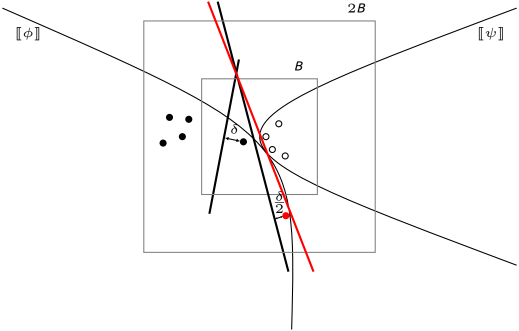
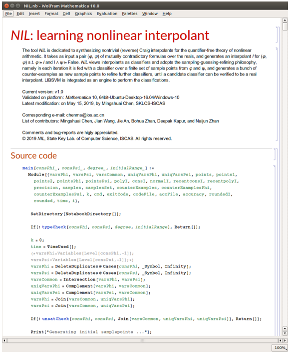


☺ The sequence of candidate interpolants converges to an actual interpolant.

Tool Support

NIL : an open-source tool in Wolfram Mathematica [2].

- LIBSVM : SVM classifications;
- Reduce : verification of candidate interpolants;
- FindInstance : generation of counterexamples;
- Rational recovery : rounding off floating-point computations [Lang, Springer NY'12].



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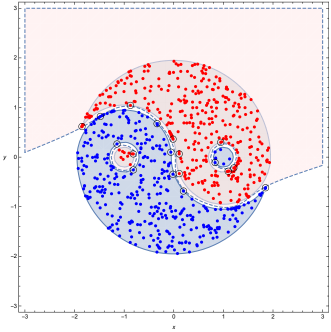
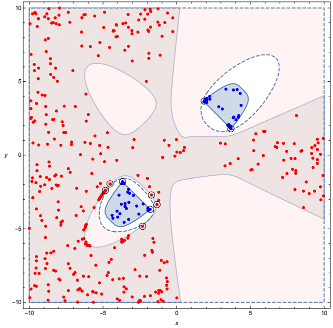
2. https://notebookarchive.org/nil-learning-nonlinear-interpolants--2021-08-5lcsyb7/

Examples

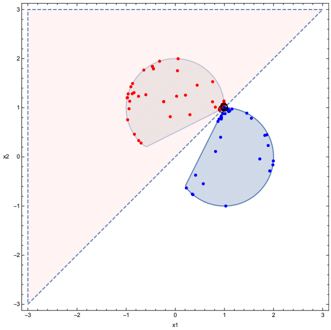
Beyond the scope of concave quadratic formulas as required in [Gan et al., IJCAR'16] :

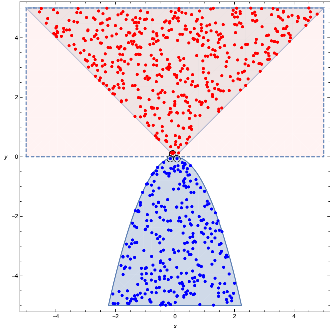




Examples

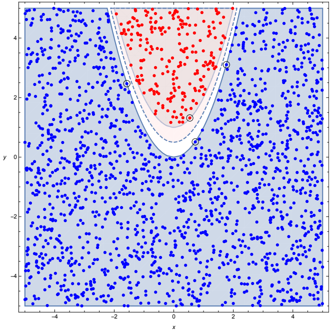
Adjacent and sharper cases as in [Okudono et al., APLAS '17] :

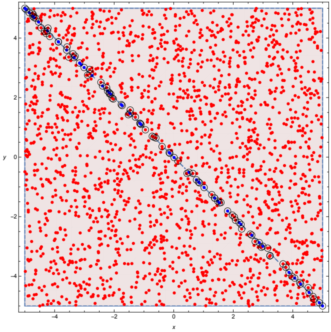




Examples

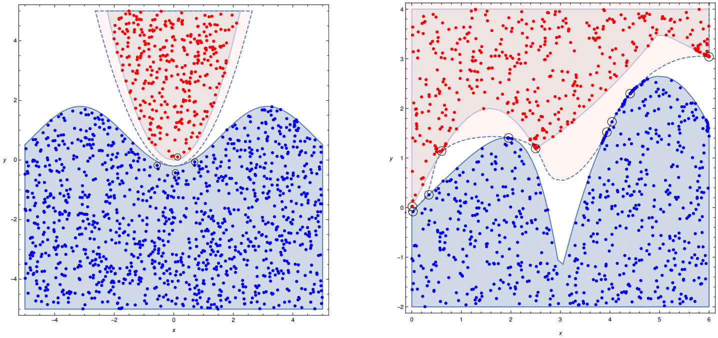
Formulas sharing parallel or coincident boundaries :





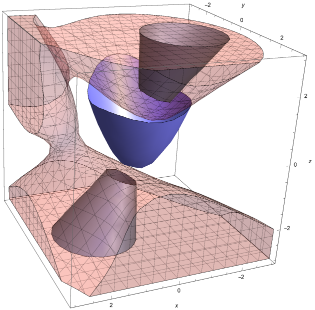
Examples

Transcendental cases from [Gao & Zufferey, TACAS '16] and [Kupferschmid & Becker, FORMATS '11], yet with simpler interpolants :

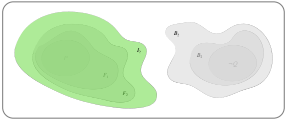


Examples

Three-dimensional case from [Dai et al., CAV '13], yet with simpler interpolants :



Summary



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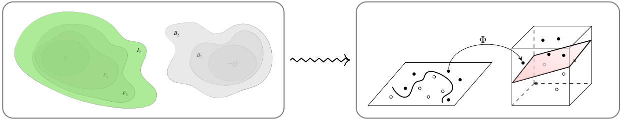
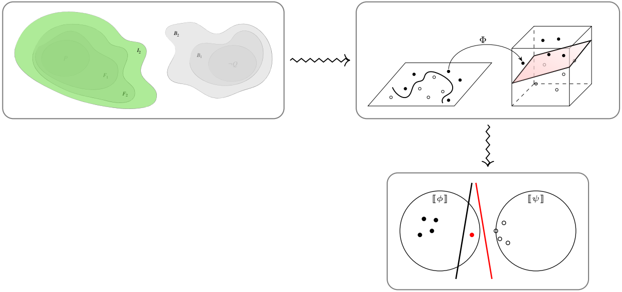
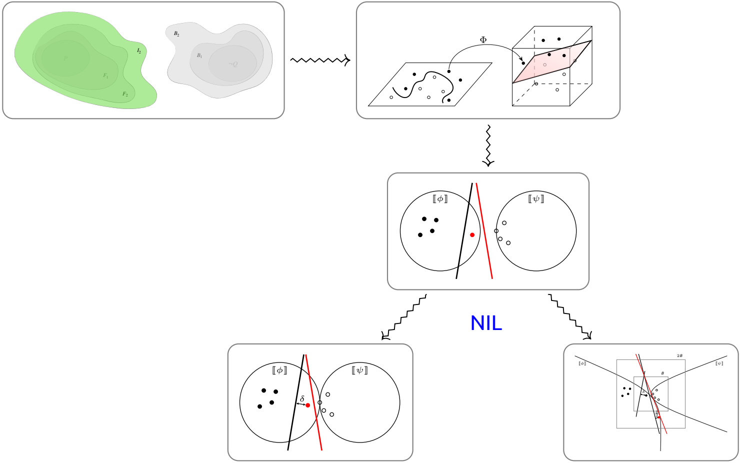


Summary



NIL

Summary



Interpolants as Loop Invariants

Example ([Sharma et al., CAV'12])

$x := 0; y := 0;$
while $(*)$
  $\{x := x + 1; y := y + 1; \}$
while $(x \neq 0)$
  $\{x := x - 1; y := y - 1; \}$
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$$A \mathrel{\widehat{=}} x_1 = 0 \wedge y_1 = 0 \wedge \mathit{ite}(b, x = x_1 \wedge y = y_1, x = x_1 + 1 \wedge y = y_1 + 1)$$

$$B \mathrel{\widehat{=}} \mathit{ite}(x = 0, x_2 = x \wedge y_2 = y, x_2 = x - 1 \wedge y_2 = y - 1) \wedge x_2 = 0 \wedge \neg(y_2 = 0)$$

$A \wedge B \models \bot$. $I(x, y) \mathrel{\widehat{=}} x = y$ s.t. $A \models I$ and $I \wedge B \models \bot$.

Interpolants as Loop Invariants

Example ([Sharma et al., CAV '12])

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$$
\begin{aligned}
A \mathrel{\hat{=}}\ & x_1 = 0 \wedge y_1 = 0 \wedge \\
& ite\,(b, \\
& \quad x = x_1 \wedge y = y_1, \\
& \quad x = x_1 + 1 \wedge y = y_1 + 1) \\
B \mathrel{\hat{=}}\ & ite\,(x = 0, \\
& \quad x_2 = x \wedge y_2 = y, \\
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& x_2 = 0 \wedge \neg(y_2 = 0)
\end{aligned}
$$

$A \wedge B \models \bot.$ $\quad$ $I(x, y) \mathrel{\hat{=}} x = y$ s.t. $A \models I$ and $I \wedge B \models \bot.$

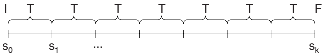

Figure – Bounded model checking.

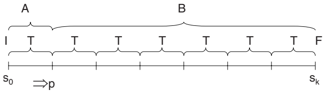

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Figure – Computing image by interpolation.

Interpolation-based Verification

☹ The bottleneck of existing formal verification techniques lies in scalability.

Interpolation-based Verification

☹ The bottleneck of existing formal verification techniques lies in scalability.

☺ Interpolation helps in scaling these verification techniques due to its inherent capability of local and modular reasoning :

- Nelson-Oppen method : equivalently decomposing a formula of a composite theory into formulas of its component theories;
- SMT : combining different decision procedures to verify programs with complicated data structures;
- Bounded model-checking : generating invariants to verify infinite-state systems due to McMillan;
- ...

Benchmark Examples

Category	ID	Name	ϕ	ψ	I	Time/s
with/without rounding	1	Dummy	$x < -1$	$x \geq 1$	$x < 0$	0.11
	2	Necklace	$y - x^2 - 1 = 0$	$y + x^2 + 1 = 0$	$-y < 0$	0.21
	3	Face	$(x+4)^2 + y^2 - 1 \leq 0 \vee$ $(x-4)^2 + y^2 - 1 \leq 0$	$x^2 + y^2 - 64 \leq 0 \wedge$ $(x+4)^2 + y^2 - 9 \geq 0 \wedge$ $(x-4)^2 + y^2 - 9 \geq 0$	$\frac{x^4}{223} - \frac{x^3 y}{356} + x^2(\frac{y^2}{45} - \frac{y}{170} - \frac{2}{9}) +$ $x(\frac{y^3}{89} + \frac{y^2}{68} - \frac{y}{74} - \frac{1}{55}) + \frac{y^4}{146} +$ $\frac{y^3}{95} + \frac{y^2}{37} + \frac{y}{366} + 1 < 0$	0.33
	4	Twisted	$x^2 - 2xy^2 + 3xz - y^2$ $- yz + z^2 - 1 \geq 0 \wedge$ $\frac{1}{120}\left(-x^6 - y^6\right) + x^2 z^2 -$ $x^2 + \frac{1}{6}\left(x^4 + 2x^2y^2 + y^4\right) +$ $y^2 z^2 - y^2 - 4 \leq 0$	$w^2 + 4(x-y)^4 + (x+y)^2 - 80 \leq 0 \wedge$ $- w^2(x-y)^4 + 100(x+y)^2 - 3000 \geq 0$	$-\frac{x^4}{160} + x^3\left(\frac{y}{170} - \frac{1}{113}\right) + x^2\left(-\frac{y^2}{225} + \frac{y}{76} + \frac{2}{27}\right) +$ $x\left(\frac{y^3}{259} + \frac{y^2}{63} + \frac{5y}{51} - \frac{1}{316}\right) - \frac{y^4}{183} - \frac{y^3}{94} + \frac{y^2}{14} + \frac{y}{255} - 1 < 0$	140.62
	5	Ultimate	$(x^2 + y^2 - 3.8025 \leq 0 \wedge y \geq 0 \vee$ $(x-1)^2 + y^2 - 0.9025 \leq 0) \wedge$ $(x-1)^2 + y^2 - 0.09 > 0 \wedge$ $(x+1)^2 + y^2 - 1.1025 \geq 0 \vee$ $(x+1)^2 + y^2 - \frac{1}{25} \leq 0$	$(-3.8025 + x^2 + y^2 \leq 0 \wedge -y \geq 0 \vee$ $- 0.9025 + (-1-x)^2 + y^2 \leq 0) \wedge$ $- 0.09 + (-1-x)^2 + y^2 > 0 \wedge$ $- 1.1025 + (1-x)^2 + y^2 \geq 0 \vee$ $- \frac{1}{25} + (1-x)^2 + y^2 \leq 0$	$\frac{x^7}{27} + x^6(-\frac{y}{5} - \frac{1}{96}) + x^5(\frac{2y^2}{9} - \frac{y}{32} - \frac{1}{2}) +$ $x^4(-\frac{2y^3}{9} + \frac{y}{3} + \frac{1}{31}) + x^3(\frac{y^4}{11} + \frac{y^3}{10} - \frac{10y^2}{13} + \frac{y}{18} + \frac{15}{16}) +$ $x^2(-\frac{y^5}{25} - \frac{y^4}{18} - \frac{y^3}{3} + \frac{y^2}{10} - \frac{1}{32}) +$ $x\left(\frac{y^6}{71} + \frac{2y^4}{11} - \frac{y^3}{25} - y^2 - \frac{y}{45} - \frac{3}{8}\right) +$ $\frac{y^6}{48} - \frac{y^5}{7} + \frac{y^4}{6} - \frac{y^3}{2} - \frac{y^2}{6} - \frac{y}{59} + \frac{1}{85} < 0$	48.82
	6	IJCAR16-1	$- x_1^2 + 4x_1 + x_2 - 4 \geq 0 \wedge$ $- x_1 - x_2 + 3 - y^2 > 0$	$-3x_1^2 - x_2^2 + 1 \geq 0 \wedge x_2 - z^2 \geq 0$	$1 - \frac{3x_1}{4} - \frac{x_2}{2} < 0$	0.16
	7	CAV13-1	$1 - a^2 - b^2 > 0 \wedge a^2 + b - 1 - x = 0 \wedge$ $b + bx + 1 - y = 0$	$x^2 - 2y^2 - 4 > 0$	$-1 + \frac{x^2}{2} - \frac{y}{3} + \frac{xy}{3} - \frac{y^2}{4} < 0$	3.25
	8	CAV13-2	$x^2 + y^2 + z^2 - 2 \geq 0 \wedge$ $1.2x^2 + y^2 + xz = 0$	$20 - 3x^2 - 4y^3 - 10z^2 \geq 0 \wedge$ $x^2 + y^2 - z - 1 = 0$	$105x^4 + x^2(140y^2 + 24y(5z+7) + 35z(3z+8)) +$ $2(70y^3 z + 5y^2(12z^2 + 21z + 28) - 14y(6z^3 + 5z^2 +$ $10) - 35(3z^4 + 8z^2 + 4z - 9)) < 14x(20x^2(z+1) +$ $10y^2(z+2) - 3y(4z^2 - 5z + 4) - 20z(z^2 + 2))$	3857.89
	9	CAV13-3	$vc < 49.61 \wedge fa = 0.5418vc^2 \wedge$ $fr = 1000 - fa \wedge ac = 0.0005fr \wedge$ $vc_1 = vc + ac$	$vc_1 \geq 49.61$	$-1 + \frac{2vc_1}{99} < 0$	40.63
with rounding	10	Parallel parabola	$y - x^2 - 1 \geq 0$	$y - x^2 < 0$	$\frac{1}{2} + x^2 < y$	4.50
	11	Parallel halfplane	$y - x - 1 \geq 0$	$y - x + 1 < 0$	$x < y$	2.46
	12	Sharper-1	$y + 1 < 0$	$x^2 + y^2 - 1 \leq 0$	$2 + y < y^2$	2.19
	13	Sharper-2	$y - x > 0 \wedge x + y > 0$	$y + x^2 <= 0$	$y > 0$	2.38
	14	Coincident	$x + y > 0 \vee x + y < 0$	$x + y = 0$	$(x+y)^2 > 0$	0.18
	15	Adjacent	$y - x^2 > 0$	$y - x^2 <= 0$	$x^2 < y$	0.25
	16	IJCAR16-2	$- y_1 + x_1 - 2 \geq 0 \wedge 2x_2 - x_1 - 1 > 0 \wedge$ $- y_1^2 - x_1^2 + 2x_1 y_1 - 2y_1 + 2x_1 \geq 0 \wedge$ $- y_2^2 - y_1^2 - x_2^2 - 4y_1 + 2x_2 - 4 \geq 0$	$- z_1 + 2x_2 + 1 \geq 0 \wedge 2x_1 - x_2 - 1 > 0 \wedge$ $- z_1^2 - 4x_2^2 + 4x_2 z_1 + 3z_1 - 6x_2 - 2 \geq 0 \wedge$ $- z_2^2 - x_1^2 - x_2^2 + 2x_1 + z_1 - 2x_2 - 1 \geq 0$	$x_1 < x_2$	12.33
	17	CAV13-4	$xa_1 + 2ya_1 \geq 0 \wedge xa_1 + 2ya_1 - x_1 = 0 \wedge$ $- 2xa_1 + ya_1 - y_1 = 0 \wedge x - x_1 - 1 = 0 \wedge$ $y = y_1 + x \wedge xa = x - 2y \wedge ya = 2x + y$	$xa + 2ya < 0$	$2xa + 4ya > 5$	3.10
beyond polynomials	18	TACAS16	$y - x^2 \geq 0$	$y + \cos x - 0.8 \leq 0$	$15x^2 < 4 + 20y$	12.71
	19	Transcendental	$\sin x \geq 0.6$	$\sin x \leq 0.4$	SVM failed	–
unbalanced	20	Unbalanced	$x > 0 \vee x < 0$	$x = 0$	$x^2 > 0$	0.11

Interpolants of Simpler Forms

Name	Interpolants by *NIL*	Interpolants from the sources
IJCAR16-1	$1 - \frac{3x_1}{4} - \frac{x_2}{2} < 0$	$-3 + 2x_1 + x_1^2 + \frac{1}{2}x_2^2 > 0$
CAV13-1	$-1 + \frac{x^2}{2} - \frac{y}{3} + \frac{xy}{3} - \frac{y^2}{4} < 0$	$436.45(x^2 - 2y^2 - 4) + \frac{1}{2} \leq 0$
CAV13-2	$105x^4 + x^2(140y^2 + 24y(5z+7) + 35z(3z+8)) + 2(70y^3z + 5y^2(12z^2 + 21z + 28) - 14y(6z^3 + 5z^2 + 10) - 35(3z^4 + 8z^2 + 4z - 9)) < 14x(20x^2(z+1) + 10y^2(z+2) - 3y(4z^2 - 5z + 4) - 20z(z^2+2))$	$-14629.26 + 2983.44x_3 + 10972.97x_3^2 + 297.62x_2 + 297.64x_2x_3 + 0.02x_2x_3^2 + 9625.61x_2^2 - 1161.80x_2^2x_3 + 0.01x_2^2x_3^2 + 811.93x_2^3 + 2745.14x_2^4 - 10648.11x_1 + 3101.42x_1x_3 + 8646.17x_1x_3^2 + 511.84x_1x_2 - 1034x_1x_2x_3 + 0.02x_1x_2x_3^2 + 9233.66x_1x_2^2 + 1342.55x_1x_2^2x_3 - 138.70x_1x_2^3 + 11476.61x_1^2 - 3737.70x_1^2x_3 + 4071.65x_1^2x_3^2 - 2153.00x1_2x_2 + 373.14x_1^2x_2x_3 + 7616.18x_1^2x_2^2 + 8950.77x_1^3 + 1937.92x_1^3x_3 - 64.07x_1^3x_2 + 4827.25x_1^4 > 0$
CAV13-3	$-1 + \frac{2vc_1}{99} < 0$	$-1.3983vc_1 + 69.358 > 0$
Sharper-1	$2 + y < y^2$	$34y^2 - 68y - 102 \geq 0$
Sharper-2	$y > 0$	$8y + 4x^2 > 0$
IJCAR16-2	$x_1 < x_2$	$-x_1 + x_2 > 0$
CAV13-4	$2xa + 4ya > 5$	$716.77 + 1326.74(ya) + 1.33(ya)^2 + 433.90(ya)^3 + 668.16(xa) - 155.86(xa)(ya) + 317.29(xa)(ya)^2 + 222.00(xa)^2 + 592.39(xa)^2(ya) + 271.11(xa)^3 > 0$
TACAS16	$15x^2 < 4 + 20y$	$y > 1.8 \vee (0.59 \leq y \leq 1.8 \wedge -1.35 \leq x \leq 1.35) \vee (0.09 \leq y < 0.59 \wedge -0.77 \leq x \leq 0.77) \vee (y \geq 0 \wedge -0.3 \leq x \leq 0.3)$

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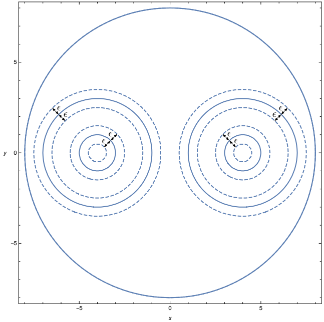
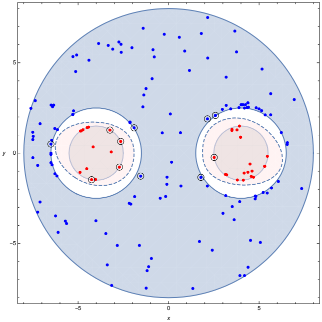
Perturbation-Resilient Interpolants



(a) ϵ-perturbations in the radii



(b) Interpolant resilient to ϵ-perturbations

Figure – Introducing ϵ-perturbations (say with ϵ up to 0.5) in ϕ and ψ. The synthesized interpolant is hence resilient to any ϵ-perturbation in the radii satisfying $-0.5 \leq \epsilon \leq 0.5$.

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Future Work : We plan to

- improve the efficiency of NIL by substituting the general purpose QE procedure with alternative methods,
- combine nonlinear arithmetic with EUFs, by resorting to, e.g., predicate-abstraction techniques,
- investigate the performance of NIL over different classification techniques, e.g., the widespread regression-based methods.

Probabilistic Craig Interpolants?

Probabilistic Craig Interpolants?

- Generalized Craig Interpolation for stochastic-SAT : resolution-based BMC of MDPs.
  - ⇒ Teige, T., Fränzle, M. : *Generalized Craig Interpolation for Stochastic Boolean Satisf. Prob.*. TACAS '11.
- Generalized Craig Interpolation for stochastic-SMT : resolution-based UMC of PHA.
  - ⇒ Mahdi, A., Fränzle, M. : *Generalized Craig Interpolation for Stochastic Satisf. Modulo Theory Prob.*. RP '14.